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Error Estimation From Three Measurement Vectors

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Consider the situation in which some phenomenon is measured at discrete points. If three independent measurements are made at each of the points then we can determine the accuracy of each of these measurements. Moreover, these three measurements can be combined into a single best estimate and its accuracy determined. This problem is of a very general nature and its solution should have wide application.

Analysis, Case 1.

Let $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$ for $i=1$ to n be three sets of measurements for some quantity $\{t_i\}$. The t_i are true but unknown values. Then let

$$x_i = t_i + \delta_i$$
$$y_i = t_i + \epsilon_i$$
$$z_i = t_i + \gamma_i \qquad \text{for } i = 1 \text{ to } n$$

We assume that the errors δ_i, ϵ_i and γ_i are random, independent and unbiased. We wish to compute the variances:

$$\hat{\sigma}_x^2 = \sum (x_i - t_i)^2/n = \sum \delta_i^2/n$$
$$\hat{\sigma}_y^2 = \sum (y_i - t_i)^2/n = \sum \epsilon_i^2/n$$
$$\hat{\sigma}_z^2 = \sum (z_i - t_i)^2/n = \sum \gamma_i^2/n$$

Define

$$P = \sum (x_i - y_i)^2$$
$$Q = \sum (x_i - z_i)^2$$
$$R = \sum (y_i - z_i)^2$$

Expanding this gives

$$P = \sum (x_i - t_i + t_i - y_i)^2 = \sum (\delta_i - \epsilon_i)^2$$
$$= \sum \delta_i^2 - 2\sum \delta_i \epsilon_i + \sum \epsilon_i^2 .$$

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The expected value of the middle term is 0 because of the assumed independence of δ_i and ϵ_i. Thus

$$P = \sum \delta_i^2 + \sum \epsilon_i^2$$

or

$$P/n = \hat{\sigma}_x^2 + \hat{\sigma}_y^2$$

Likewise

$$Q/n = \hat{\sigma}_x^2 + \hat{\sigma}_z^2$$

$$R/n = \hat{\sigma}_y^2 + \hat{\sigma}_z^2$$

or in matrix form

$$\begin{pmatrix} P/n \\ Q/n \\ R/n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{\sigma}_x^2 \\ \hat{\sigma}_y^2 \\ \hat{\sigma}_z^2 \end{pmatrix}$$

The inverse matrix gives the solution

$$\begin{pmatrix} \hat{\sigma}_x^2 \\ \hat{\sigma}_y^2 \\ \hat{\sigma}_z^2 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} P/n \\ Q/n \\ R/n \end{pmatrix}$$

or

$$\hat{\sigma}_x^2 = \frac{1}{2n}(P + Q - R)$$

$$\hat{\sigma}_y^2 = \frac{1}{2n}(P - Q + R)$$

$$\hat{\sigma}_z^2 = \frac{1}{2n}(-P + Q + R)$$

Case 2., Errors With Bias

It is possible to relax, but not eliminate, the assumption that the errors are unbiased. Now let

$$x_i = t_i + b_x + \delta_i$$

$$y_i = t_i + b_y + \epsilon_i$$

$$z_i = t_i + b_z + \gamma_i$$

We assume that b_x, b_y and b_z are the bias part of the error and that δ_i, ϵ_i and γ_i are random, independent and unbiased.

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We wish to compute the biases:

$$b_x = \sum (x_i - t_i)/n$$
$$b_y = \sum (y_i - t_i)/n$$
$$b_z = \sum (z_i - t_i)/n$$

as well as the variances:

$$\hat{\sigma}_x^2 = \sum (x_i - t_i - b_x)^2/(n-1) = \sum \delta_i^2/(n-1)$$
$$\hat{\sigma}_y^2 = \sum (y_i - t_i - b_y)^2/(n-1) = \sum \epsilon_i^2/(n-1)$$
$$\hat{\sigma}_z^2 = \sum (z_i - t_i - b_z)^2/(n-1) = \sum \gamma_i^2/(n-1)$$

Define

$$S = \sum (x_i - y_i)$$
$$T = \sum (x_i - z_i)$$
$$U = \sum (y_i - z_i)$$

Expanding this gives

$$S = \sum (x_i - t_i + t_i - y_i) = \sum (b_x + \delta_i - (b_y + \epsilon_i))$$
$$= n\, b_x + \sum \delta_i - n\, b_y - \sum \epsilon_i$$

The expected value of the two sums are zero because δ_i and ϵ_i are assumed to be unbiased. Thus

$$S/n = b_x - b_y$$

Similarly

$$T/n = b_x - b_z$$
$$U/n = b_y - b_z$$

or in matrix form

$$\begin{pmatrix} S/n \\ T/n \\ U/n \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

This matrix has rank 2, thus there is not a unique solution but many solutions. We could have anticipated this since intuitively it is clear that the addition of a constant amount to all measurements would not be detectable.

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Some physical aspect of the problem may provide a constraint which will yield a unique solution. For example, suppose that one of the measurement errors is unbiased. If say

$$b_x = 0$$

then

$$b_y = -S/n$$

$$b_z = -T/n$$

This is one possible solution. We can see from the equations that adding a constant provides additional solutions.

$$b_x = c$$

$$b_y = c - S/n$$

$$b_z = c - T/n$$

In fact all possible solutions are of this form. If there is some preconceived idea of what the biases should be then we can find the solution that is closest to them. That is, if we expect the biases to be b_x^o, b_y^o and b_z^o then we can find the solution which minimizes

$$A = (b_x - b_x^o)^2 + (b_y - b_y^o)^2 + (b_z - b_z^o)^2$$

If no better values are known for b_x^o, b_y^o and b_z^o then they can be taken as 0. Substituting gives

$$A = (c - b_x^o)^2 + (c - S/n - b_y^o)^2 + (c - T/n - b_z^o)^2$$

Differentiating with respect to c gives

$$\frac{dA}{dc} = 2(c - b_x^o) + 2(c - S/n - b_y^o) + 2(c - T/n - b_z^o)$$

Setting this equal to 0 gives the value of c which minimizes A. Thus

$$(c - b_x^o) + (c - S/n - b_y^o) + (c - T/n - b_z^o) = 0$$

or

$$c = \frac{1}{3}(S/n + T/n + b_x^o + b_y^o + b_z^o)$$

The solution then becomes

$$b_x = \frac{1}{3}(S/n + T/n + b_x^o + b_y^o + b_z^o)$$

$$b_y = \frac{1}{3}(-2S/n + T/n + b_x^o + b_y^o + b_z^o).$$

$$b_z = \frac{1}{3}(S/n - 2T/n + b_x^o + b_y^o + b_z^o)$$

Now we will compute the variances. This is done with the same procedure as in case 1. Expanding the expression for P gives

$$P = \sum (x_i - t_i + t_i - y_i)^2 = \sum (b_x + \delta_i - (b_y + \epsilon_i))^2$$

$$= n(b_x - b_y)^2 + 2(b_x - b_y)\sum(\delta_i - \epsilon_i) + \sum(\delta_i^2 - 2\delta_i\epsilon_i + \epsilon_i^2)$$

Using the assumptions that δ_i and ϵ_i are independent and unbiased gives

$$P = n(b_x - b_y)^2 + \sum \delta_i^2 + \sum \epsilon_i^2$$

or

$$\frac{P - n(b_x - b_y)^2}{n-1} = \hat{\sigma}_x^2 + \hat{\sigma}_y^2$$

Similarly

$$\frac{Q - n(b_x - b_z)^2}{n-1} = \hat{\sigma}_x^2 + \hat{\sigma}_z^2$$

$$\frac{R - n(b_y - b_z)^2}{n-1} = \hat{\sigma}_y^2 + \hat{\sigma}_z^2$$

Solving these three simultaneous equations as in case 1 gives

$$\hat{\sigma}_x^2 = \frac{1}{2(n-1)}(P + Q - R) - \frac{n}{2(n-1)}\left((b_x - b_y)^2 + (b_x - b_z)^2 - (b_y - b_z)^2\right)$$

$$\hat{\sigma}_y^2 = \frac{1}{2(n-1)}(P - Q + R) - \frac{n}{2(n-1)}\left((b_x - b_y)^2 - (b_x - b_z)^2 + (b_y - b_z)^2\right)$$

$$\hat{\sigma}_z^2 = \frac{1}{2(n-1)}(-P + Q + R) - \frac{n}{2(n-1)}\left(-(b_x - b_y)^2 + (b_x - b_z)^2 + (b_y - b_z)^2\right)$$

<u>Best Estimate of</u>

The three measurements can be combined into a single estimate of t_i by

$$\hat{t}_i = c_x(x_i - b_x) + c_y(y_i - b_y) + c_z(z_i - b_z)$$

The values of c_x, c_y and c_z which minimize the variance of the error of are known to be

$$c_x = \frac{1/\sigma_x^2}{1/\sigma_x^2 + 1/\sigma_y^2 + 1/\sigma_z^2}$$

$$c_y = \frac{1/\sigma_y^2}{1/\sigma_x^2 + 1/\sigma_y^2 + 1/\sigma_z^2}$$

$$c_z = \frac{1/\sigma_z^2}{1/\sigma_x^2 + 1/\sigma_y^2 + 1/\sigma_z^2}$$

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since the errors associated with x_i, y_i and z_i were assumed to be independent. We can now use the estimates we have computed for σ_x, σ_y and σ_z in these formulas to obtain a best estimate of t_i. The variance of $\hat{t}_i$ is

$$\sigma^2(\hat{t}_i) = \frac{1}{1/\sigma_x^2 + 1/\sigma_y^2 + 1/\sigma_z^2}$$

The following is a method of estimating the error in multitemporal registration of two LANDSAT scenes. We have tried to use the same concepts as those described in the paper "An Error Analysis of a Computer Program to Register Two Frames of LANDSAT Data" by Benjamine Seyfarth.

The major differences are:

(1) The same band is used to locate a given feature in the two LANDSAT scenes.

(2) A separate error analysis is done for each of the Cartesian coordinates.

For the greatest accuracy in locating points, the same band should be used to locate a given feature in both the base scene (twice) and the overlay scene. One reason is that some features which can be accurately located in one band are difficult to locate in another band. For example, turbid water is usually easy to distinguish in band 7 but may not be distinct in the other bands. Also, there is a band to band offset in the along-scan direction. The offsets relative to band 4 are**

band	offset in pixels
4	0
5	.09
6	.18
7	.27

Error Analysis

We will use the following notation:
For the first set of data for the base scene let

$$x_i^{b1} = x_i + \epsilon_i^{b1}, \quad i = 1 \text{ to } n$$
$$y_i^{b1} = y_i + \delta_i^{b1}$$

For the second set of data for the base scene let

$$x_i^{b2} = x_i + \epsilon_i^{b2}, \quad i = 1 \text{ to } n$$
$$y_i^{b2} = y_i + \delta_i^{b2}$$

For the set of data for the overlay scene let

$$x_i^{o} = x_i + \epsilon_i^{o} + \alpha_i, \quad i = 1 \text{ to } n$$
$$y_i^{o} = y_i + \delta_i^{o} + \beta_i$$

** Page 56 of "Manual on Characteristics of LANDSAT Computer-Compatible Tapes Produced by the EROS Data Center Digital Image Processing System"

We assume the data for the overlay scene has already been transformed to the coordinate system of the base scene. The values on the left of the equal signs are the actual data. The x_i and y_i are the true values. The various ϵ_i and δ_i are the errors caused by the person selecting the points. The α_i and β_i are the errors due to the overlay model. We assume that the errors are all random, independent and have mean 0. We assume that the ϵ_i, δ_i, α_i and β_i have variances σ_{xh}^2, σ_{yh}^2, σ_{xM}^2 and σ_{yM}^2 respectively. The σ_{xh} and σ_{yh} can be regarded as the human error, however this error also depends on the resolution and quality of the scene.

Using corresponding measurements from the base scene compute

$$\sum_{i=1}^{n} \frac{(x_i^{b1} - x_i^{b2})^2}{n} = \sum_{i=1}^{n} \frac{(\epsilon_i^{b1} - \epsilon_i^{b2})^2}{n}$$

$$= \sum \frac{(\epsilon_i^{b1})^2}{n} - 2 \sum \frac{\epsilon_i^{b1}\epsilon_i^{b2}}{n} + \sum \frac{(\epsilon_i^{b2})^2}{n}$$

$$= \sigma_{xh}^2 - 2 * 0 + \sigma_{xh}^2$$

$$= 2\sigma_{xh}^2$$

Thus

$$\sigma_{xh}^2 = \frac{1}{2} \sum_{i=1}^{n} \frac{(x_i^{b1} - x_i^{b2})^2}{n}$$

Using the average measurements from the base scene and the corresponding measurements from the overlay scenecompute

$$\sum_{i=1}^{n} \frac{\left(\frac{x_i^{b1} + x_i^{b2}}{2} - x_i^{o}\right)^2}{n} = \sum_{i=1}^{n} \frac{\left(\frac{\epsilon_i^{b1} + \epsilon_i^{b2}}{2} - \epsilon_i^{o} - \alpha_i\right)^2}{n}$$

$$= \frac{1}{4}\sum \frac{(\epsilon_i^{b1})^2}{n} + \frac{1}{4}\sum \frac{(\epsilon_i^{b2})^2}{n} + \sum \frac{(\epsilon_i^{o})^2}{n} + \sum \frac{(\alpha_i)^2}{n}$$

$$= \frac{1}{4}\sigma_{xh}^2 + \frac{1}{4}\sigma_{xh}^2 + \sigma_{xh}^2 + \sigma_{xM}^2$$

$$= 3/2\, \sigma_{xh}^2 + \sigma_{xM}^2$$

By substitution we get

$$\sigma_{xM}^2 = \sum_{i=1}^{n} \frac{\left(\frac{x_i^{b1} + x_i^{b2}}{2} - x_i^{o}\right)^2}{n} - \frac{3}{4} \sum_{i=1}^{n} \frac{(x_i^{b1} - x_i^{b2})^2}{n}$$

Similarly we can obtain

$$\sigma_{yM}^2 = \sum_{i=1}^{n} \frac{\left(\frac{y_i^{b1} + y_i^{b2}}{2} - y_i^{o}\right)^2}{n} - \frac{3}{4} \sum_{i=1}^{n} \frac{(y_i^{b1} - y_i^{b2})^2}{n}$$

Some Random Comments

Since we are attempting to determine a quantity which is less than a pixel in magnitude each feature should be located as accurately as possible. One half pixel precision or better should be attempted.

To get two independent measurements for the base scene, and use only one band, it will be necessary for the person selecting the points to forget the initial choice. The intervention of time and similar monotonous work should accomplish this.

The quantity $\sum_{i=1}^{n} \frac{(x_i^{b1} - x_i^{b2})^2}{n}$ actually is a measure of the repeatability of locating points, which may not be the same as the accuracy of locating points. The repeatability would generally be less than the accuracy.

More than one scene should be analyzed in order that we can see the variation in σ_{xM} and σ_{yM}.

A second person may be used to select points in the base scene to insure the independence of the two sets of values chosen from the base scene. This adds the complication that the quantities σ_{xh} and σ_{yh} will probably be different for the two people. I have a paper which addresses this more general problem and a copy will be sent to you.

Combining σ_{xM} and σ_{yM}

It is useful to combine σ_{xM} and σ_{yM} into a single parameter that will describe the model accuracy. Let us assume that σ_{xM} and σ_{yM} have been computed in terms of pixels. Let

l_x = pixel spacing in x direction in meters
l_y = pixel spacing in y direction in meters
$\sigma_x = \sigma_{xM} * l_x$
$\sigma_y = \sigma_{yM} * l_y$

At ESCS we have used the following simple formula

$$\sigma_M \text{ (meters)} = \sqrt{\sigma_x^2 + \sigma_y^2}$$

Another approach is to determine the radius which contains a certain percent of the data. Assuming a normal distribution, the density function is

$$\frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(x^2/\sigma_x^2 + y^2/\sigma_y^2\right)}$$

We will approximate this by

$$\frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$

where

$$\sigma = \frac{\sigma_x + \sigma_y}{2}$$

Integrating over a circle of radius R gives

$$Q \equiv \int \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)} dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r\, dr\, d\theta$$

$$= 1 - e^{-\frac{R^2}{2\sigma^2}}$$

Solving this for R gives

$$R = \sigma \sqrt{2 \ln\left(\frac{1}{1-Q}\right)}$$

or

$$R = \left(\frac{\sigma_x + \sigma_y}{2}\right) \sqrt{2 \ln\left(\frac{1}{1-Q}\right)}$$

For example, the radius which would encompass 90 percent of the data ($Q = .9$) is

$$R = \left(\frac{\sigma_x + \sigma_y}{2}\right) 2.146$$

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