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COMPOSITE ESTIMATION

by

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Some Introductory Comments

At the SRS Airlie Conference last April I discussed the question, "Should composite estimation be adopted?" Several have asked, "What is composite estimation?" and have expressed an interest in knowing more about what is involved. I deliberately did not attempt to describe composite estimation in the Airlie paper because of the limits of time. It seemed certain to raise innumerable questions which the circumstances would not permit pursuing at that time, so I tried to focus attention on the principles and some points of view that are involved.

The discussion that follows is regarded as an "opener" to stimulate thinking about composite estimation. Some numerical examples could have been worked out and included but that would have added to the length of this paper and delayed getting it out. Moreover, the nature of the subject calls for thought and consideration by all SRS statisticians. Hence, contributions in the form of examples and views pro or con are invited.

A distinction should be made between a priori information and sample information. That is, "information" emanating from past experience, common sense, general observation, subjective appraisal, etc., in contrast to evidence or information provided by a current sample. In other words, a priori information refers to the "information" that would exist at the time an estimate is made although indications from a current sample were not available. There are cases (usually at the lower levels of aggregation) where sample data are weaker than a priori judgment; e.g., when sampling error is very large owing to small sample size. In such cases it may be appropriate to bring a priori information into play in setting estimates. There are schemes for combining a priori information or judgment with sample information, but in my opinion such schemes should be considered only when sample data are weaker than the a priori information. When reading this paper it is important to keep in mind that it does not address that kind of situation, which, incidentally, is a good subject for another paper. In the approach discussed below the objective is not to avoid all use of judgment. The big question is a matter of how judgment is utilized.

Here are two additional comments before discussing composite estimation.

(1) When communicating among ourselves it is important that we do not use loosely such terms as bias, variance, standard error, root mean square error, estimator, expected value, accuracy, precision, etc. All of these terms have specific technical meaning which statisticians should be familiar with. One does not need to be a mathematical statistician to understand the concepts involved. Whether the word accuracy or precision is used may be unimportant when speaking with a nonstatistician, for example. On the other hand, misuse of the terms in technical discussions can be disconcerting or the cause of communication failure and misunderstanding. In the following discussion the most important definitions or concepts are variance of an estimate, unbiased estimate, expected value of an estimate, and target value (same as population parameter). For a review of these concepts, some readers might find an earlier paper of mine helpful. 1/

(2) The philosophy and methodology apropos to nonprobability sampling differs widely from the philosophy, methodology, or viewpoints underlying probability sampling and statistical inference. Perhaps we have been stressing probability sampling when we should have been stressing statistical inference because the latter is the reason for probability sampling. It is important that statistical inference be more fully understood.

In a sense, the Crop Reporting Board has always used composite estimation. The key point, which the Airlie paper attempted to cover, is a matter of how the composite estimation process is performed; that is, when and how subjective evaluations are used. The responsibility of the CRB for official estimates would not be changed under the approach discussed herein; but the methods of carrying out that responsibility would be changed considerably. In a few cases SRS has already made formal use of the composite estimation, notably in the labor surveys conducted for the Department of Labor and in our estimates from multiple frame research surveys.

The objective of composite estimation is to perform the estimation process: (1) as objectively as possible, (2) as accurately as possible with the aid of statistical theory, estimates of variance, and the analysis of past performance of various indications, and (3) without loss of special knowledge and experience which various individuals might have or useful information of any type. Let's look at a composite estimator and consider how the three parts of the objective come into play. Composite estimation refers to a mathematical procedure for combining information about one item into an estimate for the item.

1/ "Bases for Evaluation of Estimates," p. 46 of the Proceedings of the SRS National Conference, Blacksburg, Va., September 22-24, 1968.

A Composite Estimator

In some cases very complex mathematical formulations might be needed but the following form of a composite estimator can be used quite generally and is appropriate as an elementary form to start discussion:

$$E = \Sigma w_i f_i E_i$$

where E or the formula itself is called a composite estimator, a particular value of E is called a composite estimate;

E_i is the ith estimator corresponding to the ith indicator, a particular value of E_i is the ith estimate corresponding to the ith indication.

f_i is a factor either equal to 1 or approximately equal to 1 as discussed below,

and w_i is the weight to be given to the ith estimator. Note that $\Sigma w_i = 1$.

In some contexts, estimate and estimator are used interchangeably. For example, "unbiased estimate" and "unbiased estimator" mean the same thing. Remember that the principles involved call for setting the values of w_i and f_i before the estimates E_i are seen.

All estimators, E_i, which should receive some weight, are included in the formula regardless of source; e.g., whether probability sampling is involved or not. Ideally, the factor f_i should be 1 for all estimators; that is, we would like all estimators to be unbiased. But we know they may not all be on the same "level" (shooting at the same target). Hence, f_i is included in the composite estimator as an adjustment factor. There may be situations where bias corrections should be put in the composite estimator in some other form.

Let's assume that "T" is the target value we are estimating. (Incidentally, this does not mean that T is necessarily the ideal true value that we want to estimate--rather it is the target value which we have agreed, perhaps implicitly, to estimate.) In mathematical parlance the idea or goal is to choose f_i so the expected value of $(f_i E_i)$ is equal to T. For those estimators which are considered to be "on target" (that is, where there is no need for any adjustments for level) the values of f_i are equal to 1. Other estimators will have values of f_i associated with them which are either greater or less than one.

To illustrate, consider corn yield and assume there are two estimators-- E_1 from an objective yield survey and E_2 from farmer reports. In some areas they are not on the same level. Assume the CRB level and E_2 are in agreement which is equivalent to assuming the expected value of E_2 is equal to T. The value of f_2 is 1 and the value of f_1 might be set at .96, for example. One function of the CRB, under the approach being discussed, is to determine or agree on the values of f_i in advance.

Note that the proposed system has a built-in means of shifting from one level to another over a predetermined period of years if the CRB should decide to do so. It is a simple matter to gradually increase f_2 by predetermined increments to get to a new level and also to gradually increase f_1. There are, of course, other ways of making the change, but the system automatically has built into it a numerical record of exactly how a change of level was or will be accomplished.

Determination of Weights

Let's turn attention to the weights. First, let $E_i' = f_i E_i$ and rewrite the composite estimator as follows:

$$E = \Sigma w_i E_i'$$

At this point all of the E_i' are treated as unbiased estimates of the same target T. Biases, because of their nature, are rarely if ever known exactly. Therefore, the factors f_i cannot be determined exactly and we cannot state rigorously that the E_i are unbiased estimates of the "true value" or even of the same target. But, having done our best to determine values of f_i with the objective of having all E_i unbiased estimates of the same target, we now proceed accordingly. It is known, if the estimates E_i' are in fact all unbiased estimates of T, that the composite E is also unbiased regardless of the weights assigned to the individual estimates. (This means the composite is unbiased even though the weights are subjectively determined, provided the weights are set before the estimates are known.) The obvious criterion to apply at this point is: Select weights that will minimize the variance of the composite estimate.

It is recognized that we are not involved in an exact science, especially as many of the estimators do not originate from probability samples. Incidentally, objective means of obtaining estimates in lieu of chart reading would be in keeping with the objectives of composite estimation, but that should not preclude getting started with the use of composite estimation. The objectives or criteria for setting values of the adjustment factors f_i and of the weights w_i are quite clear. For estimators based on probability samples the basis for weights is variances of the estimators (and covariances if the estimators are correlated). Otherwise, weights are assigned arbitrarily taking into account the past performance of the particular estimators or other relevant information. In any event, as stated above, the objective in setting the weights is clear--try to minimize the variance of the composite estimate. Interplay between contributions of theory and experience will in the long run enable getting closer, and at a faster rate, to optimum weights than the weighting implicit in the present system.

Optimum weighting, like other efforts toward optimization, cannot be fully achieved. In other words, the best we can do is estimate the optimum weights. Fortunately, small departures from optimum usually have very little impact.

Adaptability

Composite estimation is very flexible and adaptable. Shifting from one level to another has been mentioned. Secondly, various pieces of information can be included in the composite estimator. In fact, the idea is to incorporate all useful bits or sources of information in the composite estimator. Suppose, for an early season forecast, one wishes to give some weight or consideration to an historical average yield adjusted for trend. The adjusted historical average can be included in the composite estimator and given any weight depending upon the amount of dampening effect on the forecast that one wants to introduce. That is, the early season forecasts would be "pulled" in the direction of the historical adjusted average. The amount of the pulling effect would be decreased as the season progresses and stronger evidence accumulates on what the outcome of the crop will be. It is cited as an illustration of the adaptability of the approach and is just one idea on how a relevant point might be brought into the estimator. It might be appropriate for some crops but not others. The objective yield estimators for forecasts already have a dampening effect built in.

There are cases where the weights should be changed from one monthly report to the next. Cotton is a good example. As is done now subjectively, the weight for the estimator based on ginners' reports could increase as the season progresses. Composite estimation calls for formalizing this procedure and setting the weights in advance. One question arises immediately. As the proportion ginned by a given date varies from year to year, shouldn't the weight for the ginner's estimator for a given month remain flexible? Assuming the answer is 'yes,' this can be accommodated in the composite estimator by replacing the weight with a mathematical expression which states the weight as a function of the proportion of the crop ginned. The weight for the other estimators would be affected. They would be changed proportionally recognizing that $\Sigma w_i = 1$.

As stated earlier, we combine data or information from several sources as a matter of everyday practice in setting an estimate. Some may believe there are situations where the process of combining indications cannot be adequately simulated mathematically. Don't misjudge the flexibility of mathematical simulation including the possibility of incorporating market check data. In some cases the mathematical representation of an estimation process will need to be complex because the process is complex; however, mathematics beyond elementary algebra is not needed to express a composite estimation process in terms of an equation or a system of equations. As in the case of livestock inventory statistics, an estimate for one item might be related to estimates for other items. That kind of situation might present some challenges on how to formulate the process mathematically, but it can be done. Composite estimation need not sacrifice accuracy in order to achieve the objectives stated at the bottom of page 2. On the contrary, accumulated experience with composite estimation and interplay between theory and practice will achieve higher levels of accuracy than is possible under the present system.

Some Results from Mathematical Statistics

The simplest case of composite estimation is the combining of two unbiased, independent estimators, E_1 and E_2 from probability sampling. The optimum weights are the reciprocals of the variances of the estimators. Without loss of generality we can assume that the variance of E_1 is smaller than the variance of E_2. In other words, let $\sigma_2^2 = k^2\sigma_1^2$ where $k \geq 1$, σ_1^2 and σ_2^2 are the variances of E_1 and E_2. In Table 1 the optimum weights W_1 and W_2 are shown in the second and third columns for the selected values of k given in the first column. The remainder of the table shows the variance of the composite estimate, when various weights are used, relative to the variance of the composite estimate weighted optimally.

To illustrate, assume the standard error (or coefficient of variation) for the second estimator is two times larger than the first; i.e., k = 2. The optimum weights are $W_1 = .8$ and $W_2 = .2$. The entries (variances) in the line for k = 2 are all greater than 1 except for the optimum point which in this case falls in the column headed $w_1 = .8$ and $w_2 = .2$. According to the table, if weights $w_1 = .6$ and $w_2 = .4$ were used instead of the optimum weights, the variance of the composite estimate would be 25 percent larger. Results in Table 1 for k = 1, 2, and 4 are shown graphically in Figure 1.

Study Table 1 and note three things:

(1) How much the weights can deviate from optimum before an appreciable increase in variance occurs (in other words, the amount of latitude that exists before an important loss in efficiency occurs). Remember the relationship between variance and sample size. To illustrate, for k = 1.5

and equal weights, observe that the entry in Table 1 is 1.17. This means that if equal weights were used instead of the optimum weights, .69 and .31, the variance of the composite estimate is 17 percent larger than optimum; or in terms of sample size the sample (or samples) would have to be increased 17 percent to offset the loss of efficiency due to failure to use optimum weights. From Figure 1 as well as Table 1, small departures from optimum may appear unimportant, except when the optimum weight for the best estimator is near 1. But, consider the cost of offsetting a loss of only 5 percent, for example. The cost of reducing variance by improving, where possible, the "statistical efficiency" of estimation methods is probably considerably less than the cost of collecting additional data that will give an equivalent reduction.

(2) The last column gives the variance of E_1 (remember E_1 has a lower variance than E_2) relative to the variance of the composite estimate optimally weighted. Note when the standard error of E_2 is more than three or four times greater than the standard error of E_1, the variance of the composite estimate, even with optimal weight, is not much less than the variance of E_1. Is the contribution of E_2 worth its cost? If the cost of E_2 was used to reduce the variance of E_1 would we be better off? Those are some of the kinds of questions that would be posed for consideration, and we would have guidance as to their resolution if we were practicing composite estimation.

(3) The table clearly shows that a composite estimator can, under some conditions, have a larger variance than the variance for the best estimator.

The same general points as discussed for two estimators hold for any number of unbiased, independent estimators--the optimum weights being the reciprocals of the variances of the respective estimators.

Let's proceed one step further and take into account the fact that two estimates derived from the same set of data are generally not independent. When some of the estimates are correlated the situation regarding weights becomes considerably more complex. For purposes of discussion let's assume two unbiased, but correlated, estimators E_1 and E_2. Again assume that E_1 has a lower variance than E_2; that is, $\sigma_2{}^2 = k^2\sigma_1{}^2$ where $k \geq 1$. Let r be the correlation between the two estimators.

Table 2 is analogous to Table 1. In fact, Table 1 is made up from the lines in Table 2 for r = o. Thus, Table 2 is simply an expansion of the picture displayed in Table 1. Note the impact on the optimum weights of correlation between the estimators, other factors being held constant. Also, note that an optimum weight can be negative. This occurs when $rk > 1$. The writer has seen a few cases in practice where an optimum weight was negative. Observe also, when rk is in the vicinity of 1, that E_2 can contribute very little. In fact, if the weights are not close to optimum it would be better to use only E_1. Table 3 illustrates this point more clearly.

In Table 3 the variance of the composite estimate is shown relative to the variance of E_1. The form of Tables 2 and 3 is the same--the only difference being that the variances in Table 3 are expressed relative to the variance of E_1 rather than to optimum. Where the variances in Table 3 are greater than 1 the variance of the composite estimate is greater than the variance of the better estimator of the two.

Table 4 shows some results for three unbiased independent estimators. Note, for example, how far equal weighting can be from optimum weighting. Also, observe when the variance of the composite is greater than the variance of E_1; that is, when the values in the column headed σ_1^2 are greater than one.

When there are several estimators, for example two derived from one sample and three from another, we may anticipate finding one or two of the estimators that contribute very little even under optimum weighting. In that case it follows that if the weights are not close to optimum an estimator in question might be causing an increase in the variance of the composite as compared to not having it.

A question to ponder: Is the value of an estimate totally accounted for in terms of its variance and the appraisal of its bias (the f factor in the composite estimator)? To state the question in another way, can a composite estimator fully utilize the contribution or value of an estimate? If the answer is 'no,' what is the value not utilized and how do you characterize the unutilized value?

Outlying Estimates

An attempt will not be made to state an exact definition of an outlying estimate. Let's say it is an estimate outside of reasonable limits in light of its variance. Hence, an estimator with a large variance will frequently give estimates that look unreasonable but are not outlying. Under the objective of minimum variance of the composite estimate, its weight will be small and appropriately determined. This means, barring arithmetic errors, that an estimate should be allowed to go into a composite even though common sense says it is inaccurate, provided it is not an outlier. Do you see a sound basis for that position? It is a position that I would not recommend if the estimate were the only one available because in that case the estimate becomes the composite and should be treated as such.

If an estimate is an outlier because of an arithmetic error or the impact of an extreme observation or two that didn't get adequately taken care of in the editing or processing of the data, a corrected value of the estimate would be put in the composite estimator and a corrected composite estimate computed. If "researching" an outlier does not give a basis for changing it, leaving purely arbitrary action as the only course, then action regarding the composite estimate is also arbitrary and whether the estimate or its weight is changed is a moot point on that occasion. However, the situation calls for reconsideration of the weights (and perhaps the adjustment factors, f) before time for the next report. Incidentally, a discussion of techniques for dealing with extreme observations would be an appropriate subject for another in-house article.

Setting the weights in advance introduces objectivity and helps to avoid personal biases in the determination of estimates. The theme of the Airlie paper was that contribution of subjective evaluations, with reference to combining information from various sources, can be exhausted in the process of setting up a composite estimator. Moreover, because of the contributions of theory and of interplay between theory and practice, pursuit of the objective of minimum variance is greatly facilitated.

After a system of composite estimation has been set up and some experience with it has occurred, a very high proportion of the composite estimates should be acceptable. However, in accordance with good statistical practice all estimates (composite or not) should be checked and reviewed. If a composite estimate appears unreasonable or totally unacceptable, some action is called for. The general point of view involved is to make changes only when arithmetic errors are found or when specific convincing reasons are surfaced which justify making a change.

Some additional questions to ponder:

(1) Barring outlying estimates, can anyone provide evidence in support of making the weight assigned to an estimate a function of how reasonable it looks? Is it possible to do better by making the weight assigned to an estimate related to how good it looks rather than keeping the weight proportional to the reciprocal of its variance? This is a key question and much can be said about it. As the question cannot be adequately treated in a few paragraphs, perhaps I will discuss it in a later communication if a general need for such discussion appears. My short generalized conclusion is quite obvious because it's a major reason for the approach to composite estimation which is recommended herein for your consideration.

(2) Do you agree that it is relatively more important to use composite estimation when the levels of accuracy involved are high, say about one percent, than when the levels of accuracy are lower, for example about five percent? This question is asked in view of our general goal of achieving higher and higher levels of accuracy. Any views on that question are of considerable interest.

Summary

To summarize, let's refer back to the objectives of composite estimation:

(1) Objectivity - As weights and any adjustment factors for level are determined before the estimates are seen, the composite estimate is more objectively determined. The word "more" is included in the previous sentence because a chart reading might be influenced by knowledge of the weight it will carry in the composite. Objective procedures for deriving estimates in lieu of subjective chart reading should receive attention. If the reading is replaced by an objective process, the composite estimate would be objectively determined. After experience with composite estimation has been acquired, it is expected that a very high proportion (95 percent or higher) of the composite estimates will be accepted.

As each CRB member has his own implicit weighting pattern, there is the possibility of inconsistency or variation (perhaps it is negligible) reflected in Board estimates owing to changes in Board membership. Under composite estimation, weights would be changed or updated only when supported by new evidence on the variances of the estimators.

Considering the scientific world in which we live and the status of statistical technology, it is increasingly difficult to justify subjective estimation procedures. The approach proposed includes discretion for rejecting or changing a composite estimate when convincing evidence is surfaced after it has been derived.

(2) Accuracy - Accuracy is a function of bias and variance. In the estimator, factors are included where needed in an effort to make all estimates unbiased estimates of the same target. Then, the criterion of selecting weights that will minimize the variance of the composite estimate is applied. Theory and variance analyses can help eliminate a lot of guesswork about what the weights should be. If estimates of variance (and covariances where needed) exist for all of the estimators, an estimate of the variance of the composite estimate is readily obtained.

(3) Judgment values - In the proposed system, value judgments or subjective appraisals continue to play a vital part but are exercised under the criteria of objectivity and minimum variance. In the long run, the writer believes the proposed system will enhance understanding of weighting and estimation procedure and will enable more accurate estimates. For example, from general experience and observation one might have concluded that a particular estimator should receive more weight than another. His judgment might be very good as far as it goes, but he can only guess empirically at the weights in the absence of the contributions of theory and variance analysis. A composite estimate would not be accepted blindly, but under

the principles involved it would be changed only when specific new evidence is surfaced after its value is known--evidence or information that had not been adequately incorporated in the composite estimator.

Composite estimation would introduce opportunity for interplay between theory and practice, an important factor governing rate of progress toward achievement of highest possible levels of accuracy. It would undoubtedly provoke much debate and exchange of views regarding the quality or accuracy of various sources of information. The CRB should foster such discussion and specify the mechanism for deciding which estimators should be included in a composite estimator and what the weights and values of f_i should be. This should not be viewed as an onerous task. Such dialogue, entered into in the right spirit, can stimulate much constructive thinking, lead to better understanding, and be a very progressive force.

Table 1.- Variance of the composite estimate relative to optimum

k	Optimum weights W_1	Optimum weights W_2	Variance[1] of the composite estimate when $w_1=.4$ $w_2=.6$	$w_1=.5$ $w_2=.5$	$w_1=.6$ $w_2=.4$	$w_1=.7$ $w_2=.3$	$w_1=.8$ $w_2=.2$	$w_1=.9$ $w_2=.1$	$w_1=1.0$ $w_2=0.0$
1	.50	.50	1.04	1.00	1.04	1.16	1.36	1.64	2.00
1.5	.69	.31	1.40	1.17	1.04	1.00	1.05	1.20	1.44
2.0	.80	.20	2.00	1.56	1.25	1.06	1.00	1.06	1.25
3.0	.90	.10	3.78	2.78	2.00	1.44	1.11	1.00	1.11
4.0	.94	.06	6.30	4.52	3.10	2.01	1.36	1.03	1.06

1/ Note:

Variances computed are $\frac{w_1^2\sigma_1^2+w_2^2\sigma_2^2}{W_1^2\sigma_1^2+W_2^2\sigma_2^2} = \frac{w_1^2+w_2^2k^2}{W_1^2+W_2^2k^2}$

where $\sigma_2=k\sigma_1$

Optimum weights are $W_1= \frac{k^2}{1+k^2}$ $W_2= \frac{1}{1+k^2}$

w_1 and w_2 are arbitrarily choosen weights

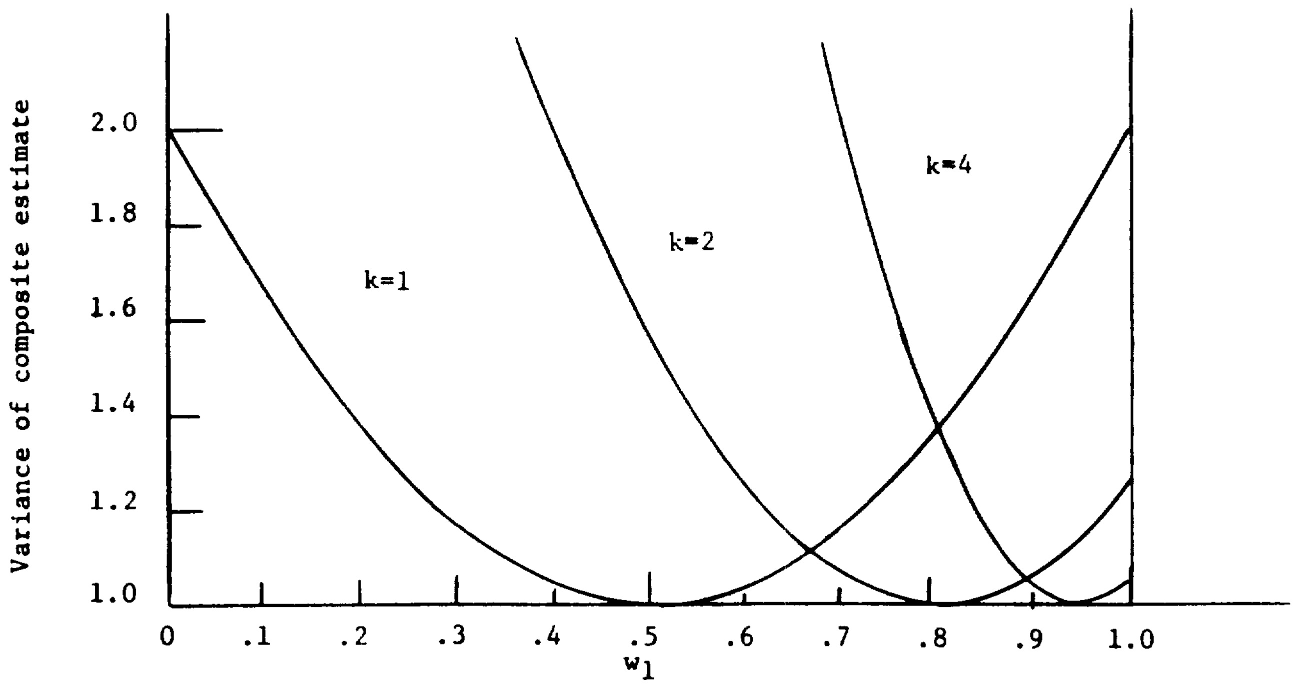

Figure 1.- Variance of E as a function of w_1

Table 2.- Variance of the composite estimate relative to optimum

k	r	Optimum weights		Variance[1] of composite estimate when assigned weights are								
		W_1	W_2	w_1=.4 w_2=.6	w_1=.5 w_2=.5	w_1=.6 w_2=.4	w_1=.7 w_2=.3	w_1=.8 w_2=.2	w_1=.9 w_2=.1	w_1=1 w_2=0	w_1=1.1 w_2=-.1	w_1=1.2 w_2=-.2
1.0	-.4	.500	.500	1.09	1.00	1.09	1.37	1.84	2.49	3.33		
1.0	-.2	.500	.500	1.06	1.00	1.06	1.24	1.54	1.96	2.50		
1.0	.0	.500	.500	1.04	1.00	1.04	1.16	1.36	1.64	2.00		
1.0	.2	.500	.500	1.03	1.00	1.03	1.10	1.24	1.43	1.67		
1.0	.4	.500	.500	1.02	1.00	1.02	1.07	1.15	1.27	1.43		
1.0	.6	.500	.500	1.01	1.00	1.01	1.04	1.09	1.16	1.25		
1.5	-.4	.640	.360	1.60	1.20	1.02	1.04	1.27	1.70	2.35		
1.5	-.2	.662	.338	1.47	1.18	1.03	1.01	1.13	1.39	1.78		
1.5	.0	.692	.308	1.40	1.17	1.04	1.00	1.05	1.20	1.44		
1.5	.2	.736	.264	1.37	1.18	1.06	1.00	1.01	1.09	1.23		
1.5	.4	.846	.154	1.36	1.20	1.09	1.02	1.00	1.02	1.08		
1.5	.6	.931	.069	1.41	1.27	1.16	1.08	1.03	1.00	1.01		
2.0	-.4	.727	.273	2.39	1.67	1.21	1.01	1.06	1.39	1.96		
2.0	-.2	.759	.241	2.13	1.59	1.22	1.03	1.02	1.18	1.51		
2.0	.0	.800	.200	2.00	1.56	1.25	1.06	1.00	1.06	1.25		
2.0	.2	.857	.143	1.96	1.59	1.30	1.11	1.02	1.01	1.09		
2.0	.4	.941	.059	2.00	1.67	1.40	1.20	1.07	1.01	1.01	1.09	1.23
2.0	.6	1.077	-.077	2.21	1.88	1.60	1.37	1.20	1.08	1.02	1.00	1.04
3.0	-.4	.823	.177	4.63	3.11	2.01	1.30	1.01	1.12	1.64	2.56	3.90
3.0	-.2	.889	.111	3.97	2.81	1.93	1.34	1.03	1.01	1.28	1.83	2.67
3.0	.0	.900	.100	3.78	2.78	2.00	1.44	1.11	1.00	1.11	1.44	2.00
3.0	.2	.954	.046	3.76	2.85	2.13	1.58	1.21	1.03	1.02	1.19	1.54
3.0	.4	1.026	-.026	4.00	3.12	2.39	1.81	1.39	1.12	1.01	1.04	1.23
3.0	.6	1.125	-.125	4.74	3.78	2.96	2.28	1.75	1.36	1.11	1.00	1.04
4.0	-.4	.871	.129	7.75	5.19	3.24	1.89	1.15	1.03	1.50	2.59	4.28
4.0	-.2	.903	.097	6.70	4.66	3.07	1.93	1.24	1.00	1.21	1.87	2.98
4.0	.0	.941	.059	6.29	4.52	3.10	2.05	1.36	1.03	1.06	1.46	2.21
4.0	.2	.987	.013	6.32	4.66	3.31	2.27	1.54	1.12	1.00	1.20	1.70
4.0	.4	1.044	-.044	6.87	5.18	3.79	2.67	1.84	1.29	1.03	1.05	1.35
4.0	.6	1.115	-.115	8.42	6.49	4.85	3.50	2.44	1.67	1.19	1.00	1.10

1/ Note:

Variances computed are $$\frac{w_1^2\sigma_1^2+w_2^2\sigma_2^2+2w_1w_2r\sigma_1\sigma_2}{W_1^2\sigma_1^2+W_2^2\sigma_2^2+2W_1W_2r\sigma_1\sigma_2} = \frac{w_1^2+w_2^2k^2+2w_1w_2rk}{W_1^2+W_2^2k^2+2W_1W_2rk}$$

Optimum weights are $$W_1 = \frac{k^2-rk}{1+k^2-2rk} \qquad W_2 = \frac{1-rk}{1+k^2-2rk}$$

r = correlation between the estimators.
Other expressions are the same as defined previously.

Table 3.- Variance of the composite estimate relative to the variance of E_1

k	r	Optimum weights W_1	Optimum weights W_2	Variance[1/] of composite estimate when assigned weights are: Optimum	w_1=.4 w_2=.6	w_1=.5 w_2=.5	w_1=.6 w_2=.4	w_1=.7 w_2=.3	w_1=.8 w_2=.2	w_1=.9 w_2=.1	w_1=1 w_2=0	w_1=1.1 w_2=-.1	w_1=1. w_2=-.
1.0	-.4	.500	.500	.30	.33	.30	.33	.41	.55	.75	1.00		
1.0	-.2	.500	.500	.40	.42	.40	.42	.50	.62	.78	1.00		
1.0	.0	.500	.500	.50	.52	.50	.52	.58	.68	.82	1.00		
1.0	.2	.500	.500	.60	.62	.60	.62	.66	.74	.86	1.00		
1.0	.4	.500	.500	.70	.71	.70	.71	.75	.80	.89	1.00		
1.0	.6	.500	.500	.80	.81	.80	.81	.83	.87	.93	1.00		
1.5	-.4	.640	.360	.42	.68	.51	.43	.44	.54	.72	1.00		
1.5	-.2	.662	.338	.56	.83	.66	.58	.57	.63	.78	1.00		
1.5	.0	.692	.308	.69	.97	.81	.72	.69	.73	.83	1.00		
1.5	.2	.736	.264	.82	1.11	.96	.86	.82	.83	.89	1.00		
1.5	.4	.846	.154	.92	1.26	1.11	1.01	.94	.92	.94	1.00		
1.5	.6	.931	.069	.99	1.40	1.26	1.15	1.07	1.02	.99	1.00		
2.0	-.4	.727	.273	.51	1.22	.85	.62	.51	.54	.71	1.00		
2.0	-.2	.759	.241	.66	1.41	1.05	.81	.68	.67	.78	1.00		
2.0	.0	.800	.200	.80	1.60	1.25	1.00	.85	.80	.85	1.00		
2.0	.2	.857	.143	.91	1.79	1.45	1.19	1.02	.93	.92	1.00		
2.0	.4	.941	.059	.99	1.98	1.65	1.38	1.19	1.06	.99	1.00	1.07	1.22
2.0	.6	1.077	-.077	.98	2.18	1.85	1.58	1.35	1.18	1.07	1.00	.99	1.02
3.0	-.4	.823	.173	.61	2.82	1.90	1.22	.80	.62	.68	1.00	1.56	2.38
3.0	-.2	.889	.111	.78	3.11	2.20	1.51	1.05	.81	.79	1.00	1.43	2.09
3.0	.0	.900	.100	.90	3.40	2.50	1.80	1.30	1.00	.90	1.00	1.30	1.80
3.0	.2	.954	.046	.98	3.69	2.80	2.09	1.55	1.19	1.01	1.00	1.17	1.51
3.0	.4	1.026	-.026	.99	3.98	3.10	2.38	1.80	1.38	1.12	1.00	1.04	1.22
3.0	.6	1.125	-.125	.90	4.26	3.40	2.66	2.06	1.58	1.22	1.00	.90	.94
4.0	-.4	.871	.129	.67	5.15	3.45	2.15	1.26	.77	.68	1.00	1.72	2.85
4.0	-.2	.903	.097	.83	5.54	3.85	2.54	1.59	1.02	.83	1.00	1.55	2.46
4.0	.0	.941	.059	.94	5.92	4.25	2.92	1.93	1.28	.97	1.00	1.37	2.08
4.0	.2	.987	.013	1.00	6.30	4.65	3.30	2.27	1.54	1.11	1.00	1.19	1.70
4.0	.4	1.044	-.044	.97	6.69	5.05	3.69	2.60	1.79	1.26	1.00	1.02	1.31
4.0	.6	1.115	-.115	.84	7.07	5.45	4.07	2.94	2.05	1.40	1.00	.84	.93

1/ Note:

Variances recorded are $\frac{w_1^2\sigma_1^2+w_2^2\sigma_2^2+2w_1w_2r\sigma_1\sigma_2}{\sigma_1^2} = w_1^2+w_2^2k^2+2w_1w_2rk$

Table 4.- Variances of a composite of three independent estimators

1/	w_1	w_2	w_3	Variance of composite relative to: Optimum	Variance of composite relative to: σ_1^2
			$\sigma_2 = \sigma_1$, $\sigma_3 = \sigma_1$		
1	.34	.33	.33	1.00	.33
2	.34	.33	.33	1.00	.33
3	.50	.50	.00	1.50	.50
4	.60	.20	.20	1.32	.44
5	.50	.40	.10	1.26	.42
6	.50	.30	.20	1.14	.38
7	.40	.40	.20	1.08	.36
			$\sigma_2 = \sigma_1$, $\sigma_3 = 2\sigma_1$		
1	.44	.44	.11	1.00	.44
2	.34	.33	.33	1.52	.66
3	.5	.5	.0	1.15	.50
4	.5	.4	.1	1.03	.45
5	.5	.3	.2	1.15	.50
6	.4	.4	.2	1.10	.48
7	.4	.3	.3	1.40	.61
8	.3	.3	.4	1.88	.82
			$\sigma_2 = \sigma_1$, $\sigma_3 = 4\sigma_1$		
1	.48	.48	.03	1.00	.48
2	.34	.33	.33	4.15	1.97
3	.5	.5	.0	1.05	.50
4	.6	.3	.1	1.28	.61
5	.5	.4	.1	1.20	.57
6	.5	.3	.2	2.06	.98
7	.4	.4	.2	2.02	.96
8	.4	.3	.3	3.56	1.69

1/ Line 1 - optimum weights.
Line 2 - equal weights.
Line 3 - equal weights to E_1 and E_2 ignoring E_3.
Lines 4, etc. - arbitrarily selected weights.

Table 4 continued on next page -

Table 4.- Variances of a composite of three independent estimators (-continued)

1/	w_1	w_2	w_3	Variance of composite relative to: Optimum	Variance of composite relative to: σ_1^2
	$\sigma_2 = 2\sigma_1$, $\sigma_3 = 2\sigma_1$				
1	.66	.17	.17	1.00	.67
2	.34	.33	.33	1.49	.99
3	.5	.5	.0	1.88	1.25
4	.7	.2	.1	1.04	.69
5	.6	.2	.2	1.02	.68
6	.5	.4	.1	1.40	.93
7	.5	.3	.2	1.16	.77
8	.4	.4	.2	1.44	.96
9	.4	.3	.3	1.32	.88
10	.2	.4	.4	1.98	1.32
	$\sigma_2 = 2\sigma_1$, $\sigma_3 = 4\sigma_1$				
1	.76	.19	.05	1.00	.75
2	.34	.33	.33	3.08	2.30
3	.5	.5	.0	1.67	1.25
4	.9	.1	.0	1.14	.85
5	.8	.2	.0	1.07	.80
6	.8	.1	.1	1.12	.84
7	.7	.1	.2	1.57	1.17
8	.6	.2	.2	1.55	1.16
9	.5	.3	.2	1.67	1.25
10	.4	.3	.3	2.62	1.96
	$\sigma_2 = 4\sigma_1$, $\sigma_3 = 4\sigma_1$				
1	.89	.05	.05	1.00	.87
2	.34	.33	.33	4.13	3.60
3	.5	.5	.0	4.87	4.25
4	.8	.1	.1	1.10	.96
5	.7	.2	.1	1.48	1.29
6	.6	.2	.2	1.88	1.64
7	.5	.3	.2	2.67	2.33
8	.4	.3	.3	3.49	3.04
9	.2	.4	.4	5.92	5.16